SOME REMARKS CONCERNING THE STABILITY OF A LINEAR STOCHASTIC SYSTEM

(NEKOTORYE ZAMECHANIIA OTNOSITEL'NO USTOICHIVOSTI LINEINOI STOKHASTICHESKOI SISTEMY)

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The method for investigating the asymptotic stability in the mean square of a linear stochastic system with Gaussian "white noises" proposed in [1] is developed. Section 1 contains an interpretation of the result obtained [1] in terms of the stability quality criterion employed in controlled systems theory. Section 2 contains the necessary and sufficient conditions of asymptotic stability in the mean square in the Routh-Hurwitz form of a linear stochastic system for the case where the intensities of noises acting in different directions are proportional to each other. This result generalizes the stability conditions given in [1 and 2] (*).

1. Let us be given an nth order ordinary differential equations with constant coefficients

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0 \tag{1.1}$$

which after the substitution

$$y = X_1, \qquad y' = X_2, \ldots, y^{(n-1)} = X_n \tag{1.2}$$

becomes the system

$$dX_1 = X_2\, dt, \quad dX_2 = X_3\, dt, \ldots, dX_n = -\sum_{i=1}^{n} a_i X_{n-i+1}\, dt \tag{1.3}$$

Let us suppose that system (1.3) is asymptotically stable.

In the theory of controlled systems, each nonnegatively defined quadratic form

$$a(x) = \sum_{i,j=1}^{n} a_{ij} x_{n-i+1} x_{n-j+1}$$

is associated (e.g. see [3 and 4] with an integral criterion of the stability quality of the solution $X^x(t)$ of system (1.3) with the initial condition $X^x(0) = x$

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*) We take this opportunity to note that [1] contains the following misprints which make its reading difficult. In the right-hand side of Formula (2.5) on p. 406, the exponent of λ ought to contain the number 2. On p.407, fourth line from top, one should read $(-1)^{j-1}\lambda^{i+j-2}$ instead of $(-1)^{i+j-1}\lambda^{i-2}$

$$I_a(x)=\int_0^\infty a(X^x(t))\,dt \tag{1.4}$$

Along with (1.1), let us consider the stochastic system

$$y^{(n)}+[a_1+\eta_1^{\cdot}(t)]\,y^{(n-1)}+\ldots+[a_n+\eta_n^{\cdot}(t)]\,y=0 \tag{1.5}$$

Here the Gaussian "white noises" $\eta_1^{\cdot}(t),\ldots,\eta_n^{\cdot}(t)$ have a zero mathematical expectation, but can in general be dependent, so that

$$M\eta_i^{\cdot}(t)\,\eta_j^{\cdot}(s)=2a_{ij}\delta\,(t-s)$$

From the results of [1] it follows that system (1.5) is asymptotically stable in the mean square if and only if the criterion of the stability quality of the solution $X^x(t)$ of determinate system (1.1) with the initial condition $X^x(0)=\delta_{in}$ satisfy the condition

$$I_a(0,\ldots,0,1)<{}^1/_2 \tag{1.6}$$

From (1.2) and (1.4) for $x=(0,\ldots,0,1)$ we have

$$I_a(x)=\int_0^\infty\sum_{i,j=1}^n a_{ij}X_{n-i+1}^x(t)\,X_{n-j+1}^x(t)\,dt=\sum_{i,j=1}^n a_{ij}\int_0^\infty y_0^{(n-i)}(t)\,y_0^{(n-j)}(t)\,dt$$

Here $(y_0,y_0',\ldots,y_0^{(n-1)})$ is the solution of Equation (1.1) with the initial condition $(0,\ldots,0,1)$.

Integrating by parts, we readily see that with odd $i+j$,

$$\int_0^\infty y^{(n-i)}(t)\,y^{(n-j)}(t)\,dt=0$$

so that stability condition (1.6) includes only those correlation coefficients a_{ij} for which the sum $i+j$ is even. This fact was also noted in [1].

We note that the direction $(0,\ldots,0,1)$ in the space $(x_1,x_2,\ldots,x_n)$ is a special one, since it is precisely in this direction that diffusion occurs (see Formula (1.4) in [1]).

2. Now let us have the general linear stochastic system

$$X_i^{\cdot}=\sum_{j=1}^n[b_{ij}+\eta_{ij}^{\cdot}(t)]\,X_j \tag{2.1}$$

where, as before, the noises $\eta_{ij}^{\cdot}(t)$ have a zero mathematical expectation, and

$$M\eta_{ik}^{\cdot}(t)\,\eta_{jl}^{\cdot}(s)=2a_{kl}^{ij}\,\delta\,(t-s)$$

Investigation of system (2.1) involves considerable difficulties, and it appears that the necessary and sufficient conditions of stability in the mean square generally cannot be given in sufficiently effective form in terms of the parameters b_{ij}, a_{kl}^{ij}.

We shall assume that

$$a_{kl}^{ij}=c_{ij}a_{kl} \tag{2.2}$$

i.e. that the correlation matrices of the noises $\eta_{ik}^{\cdot}$ $(k=1,\ldots,n)$ and η_{jl} $(l=1,\ldots,n)$, acting in the directions x_i and x_j, respectively, are proportional to the same matrix $\|a_{kl}\|$. It is clear that the quadratic forms

$$a(x)=\sum_{k,l=1}^n a_{kl}x_kx_l,\qquad \sum_{i,j=1}^n c_{ij}x_ix_j$$

are here nonnegative definite.

Converting, as before, from the noises $\eta_{ij}^{\cdot}(t)$ to the independent "white noises", we can write Equations (2.1) in the form of a system of Ito stochastic differential equations described by the operator

$$L = L_0 + a(x) \sum_{i,j=1}^{n} c_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

Here L_0 is the Liapunov operator.

As we know, the determinate system

$$X_i^{\cdot} = \sum_{j=1}^{n} b_{ij} X_j \qquad (2.3)$$

is asymptotically stable if and only if the Routh-Hurvwitz conditions

$$\Delta_1 > 0, \qquad \Delta_2 > 0, \ldots, \Delta_n > 0$$

are fulfilled (*).

Let us first suppose that the quadratic form $a(x)$ is positive definite. Then, proceeding from Theorem 3.2 of [5], we can prove (as was done in [1]) the following statement: system (2.1) is asymptotically stable in the mean square if and only if there exists a positive definite quadratic form

$$V(x) = \sum_{i,j=1}^{n} d_{ij} x_i x_j$$

for which

$$L_0 V(x) = -a(x), \qquad \sum_{i,j=1}^{n} c_{ij} d_{ij} < \frac{1}{2}$$

Further, it is shown in [6] that the coefficient d_{ij} of the form $V(x)$ can be written as

$$d_{ij} = \frac{1}{2\Delta_n} \sum_{r=0}^{n-1} q_{ij}^{(r)} \Delta^{1,r+1}$$

where $\Delta^{1,r+1}$ is the algebraic complement of the element of the first row and $(r+1)$-th column of the last Hurwitz determinant Δ_n, while the numbers $q_{ij}^{(r)} (r = 0, 1, \ldots, n-1)$ are related to the coefficients a_{kl} by Formula

$$(-1)^n \sum_{k,l=1}^{n} a_{kl} D_{ik}(\lambda) D_{jl}(-\lambda) = \sum_{r=0}^{n-1} q_{ij}^{(r)} \lambda^{2(n-r-1)} \qquad (2.4)$$

Here $D_{ik}(\lambda)$ is the algebraic complement of the element in the ith row and jth column of the determinant of system (2.3).

Thus, system (2.1) is asymptotically stable in the mean square if and only if the following conditions are fulfilled:

$$\Delta_1 > 0, \qquad \Delta_2 > 0, \ldots, \Delta_n > 0, \qquad \Delta_n > \sum_{i,j=1}^{n} c_{ij} \Delta_{ij} \qquad (2.5)$$

where the determinant Δ_{ij} is obtained from the last Hurwitz determinant Δ_n by replacing its first row by the row $(q_{ij}^{(1)}, q_{ij}^{(2)}, \ldots, q_{ij}^{(n-1)})$ and where the numbers $q_{ij}^{(r)}$ are determined from the polynomials (2.4).

We shall show that conditions (2.4) and (2.5) remain valid even without the assumption that the form $a(x)$ is nondegenerate. To do this, along with system (2.1) we consider the other linear system

$$X_i^{\cdot} = \sum_{j=1}^{n} [b_{ij} + \eta_{ij}(t) + \zeta_{ij}(t)] X_j \qquad (2.6)$$

Here the "white noises" ζ_{ij} $(i, j = 1, \ldots, n)$ average out to zero, and independent of the totality of $\eta_{ij}^{\cdot}$, and, furthermore

$$M\zeta_{ik}^{\cdot}(t)\, \zeta_{jl}^{\cdot}(s) = 2\varepsilon^2 c_{ij}\, \delta_{kl} \delta(t-s) \qquad (\varepsilon \text{ is a small parameter})$$

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*) The construction of the determinants Δ_i, whose elements are the coefficients of the characteristic equation of system (2.3), is described in [1].

The generating operator of system (2.6) is given by Formulas

$$L_\varepsilon = L_0 + a_\varepsilon(x) \sum_{i,j=1}^{n} c_{ij} \frac{\partial^2}{\partial x_i \partial x_j}, \qquad a_\varepsilon(x) = a(x) + \varepsilon^2 \sum_{i=1}^{n} x_i^2$$

Since $a_\varepsilon(x)$ is positive definite, the conditions of asymptotic stability in the mean square of system (2.6) are of the form

$$\Delta_1 > 0, \quad \Delta_2 > 0, \ldots, \Delta_n > 0, \qquad \Delta_n > \sum_{i,j=1}^{n} c_{ij}\Delta_{ij}^{(\varepsilon)}$$

(the determinants $\Delta_{ij}^{(\varepsilon)}$ are constructed in the obvious way).

Further, representing the Liapunov function in system (2.3) in terms of its fundamental system of solutions $X^\circ_{sj}(t)$ with the initial condition $X^\circ_{sj}(0) = \delta_{sj}$ (see [7]), we can readily see that

$$\Delta_{ij}^{(\varepsilon)} = \Delta_{ij} + \varepsilon^2 \sum_{k=1}^{n} \int_0^\infty X^\circ_{ki}(t)\, X^\circ_{kj}(t)\, dt, \quad \text{or} \quad \sum_{i,j=1}^{n} c_{ij}\Delta_{ij}^{(\varepsilon)} > \sum_{i,j=1}^{n} c_{ij}\Delta_{ij} \qquad (2.7)$$

Inequality (2.7) and Theorems 5.1 and 5.2 of [5] imply that conditions (2.4) and (2.5) are necessary and sufficient for the stability of system (2.1) even if $a(x) \geq 0$.

The determinants Δ_i, Δ_{ij} are readily computable by the method given in [8 and 9].

In conclusion we note that condition (2.2) is fulfilled, specifically, in the case where all of the $\eta_{ij}{}^\cdot(t) \equiv 0$ with the exception of some single $\eta_{kl}{}^\cdot(t)$. Precisely such a case, but under more restricted assumptions as regards system (2.3), is considered in [2].

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